Kelvin Transformations
for Simulations on Infinite Domain

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Team 1 Paper Presentation
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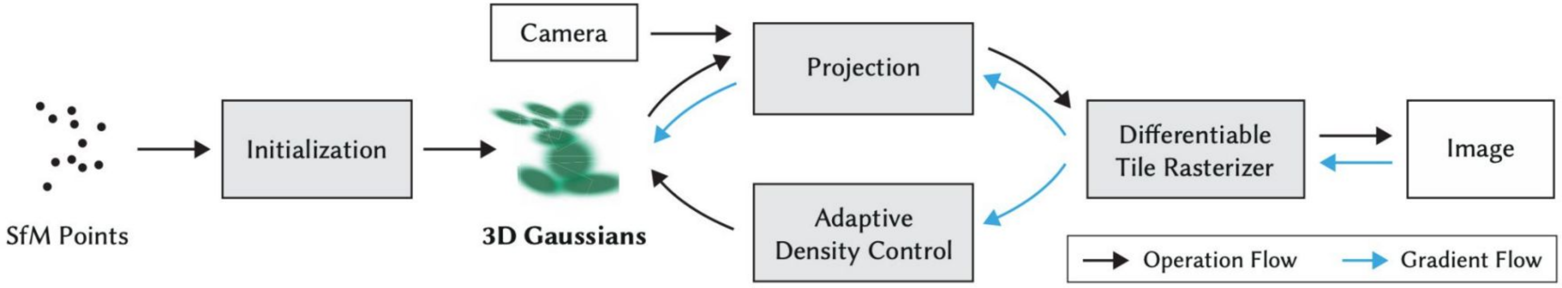
Summary of Last Presentation

3D Gaussian Splatting for Real-Time Radiance Field Rendering (SIGGRAPH 2023)



- Parametrize scene via 3D gaussians.
- Exploit existing rasterization process for fast training and testing time

Overview

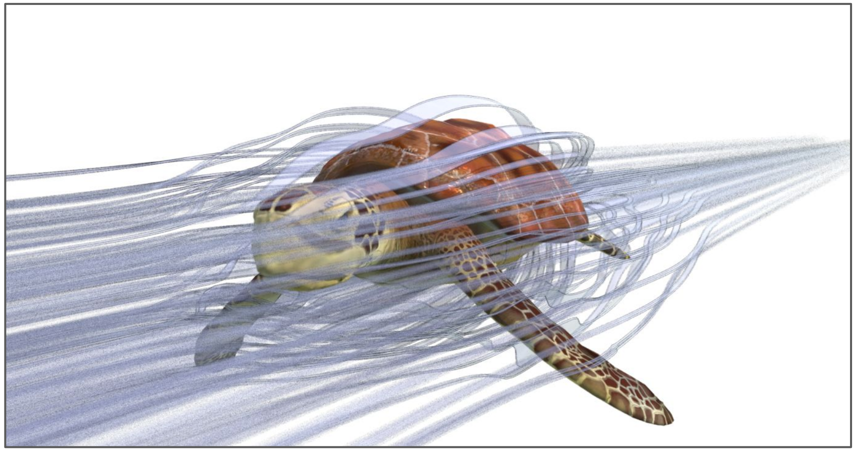
1. **Motivation**
2. **Method**
3. **Experiments**
4. **Takeaways**

Motivation

Why solving PDE in Infinite Domain?

Current Finite Element solver requires finite domain for discretization.

This means, unbounded simulation need additional **radiation condition** (boundary value at infinity) or an **biased estimate** of boundary values.

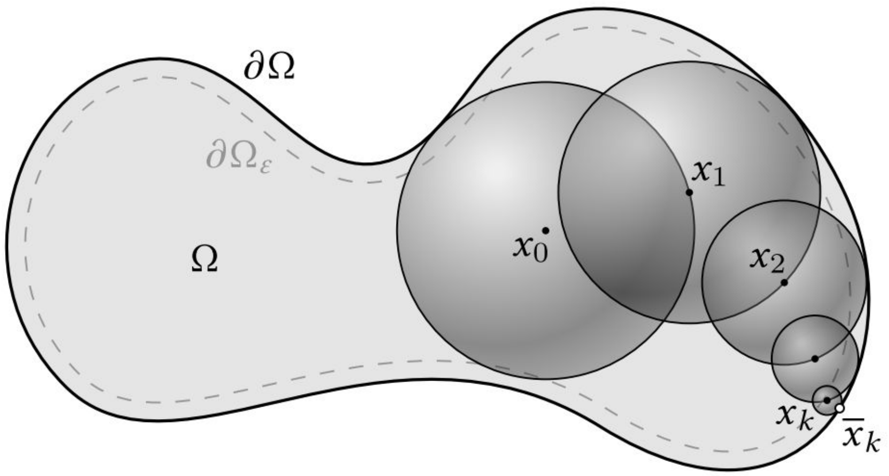


Why solving PDE in Infinite Domain?

Walk-on-Sphere

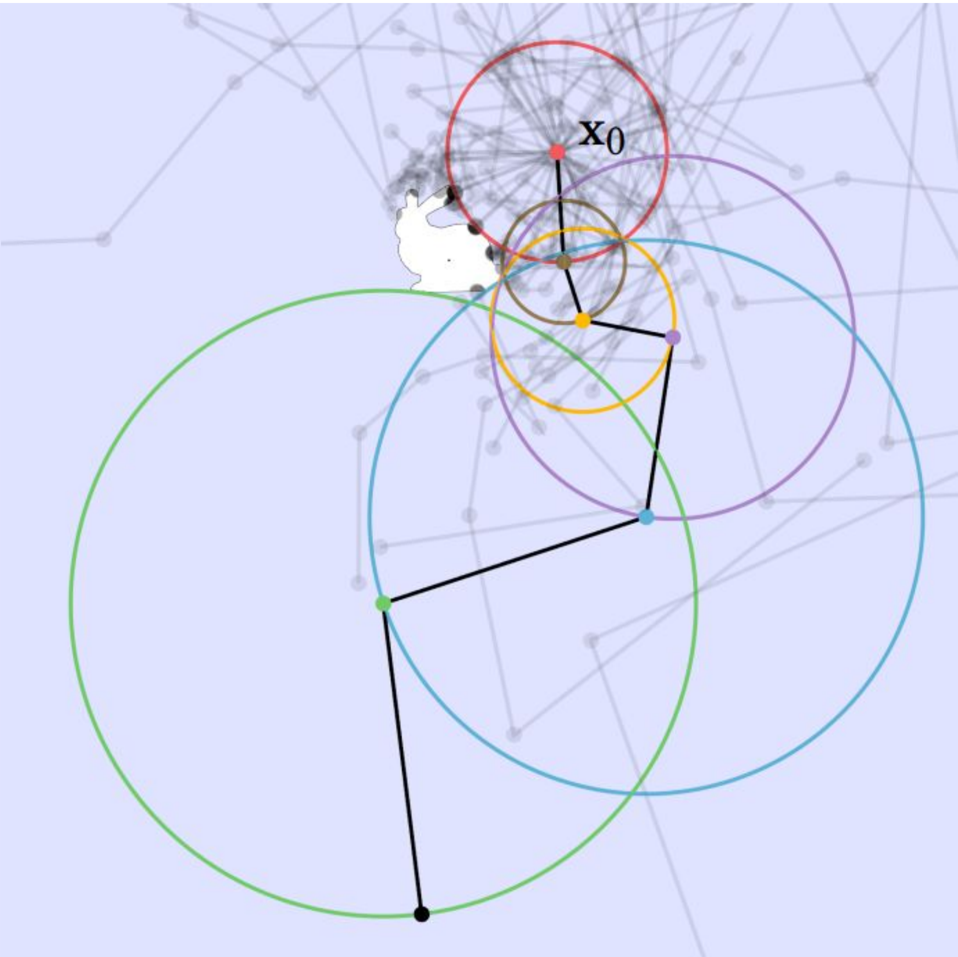
Walk on sphere sampling by choosing a random point on the biggest sphere that can fit in boundary.

1. Initialize: $x^{(0)} = x$
2. While $\text{distance}(x^{(n)} > \Gamma) > \varepsilon$
   a. Set $r_n = \text{distance}(x^{(n)}, \Gamma)$
   b. Sample γ_n uniformly at the sphere centered in $x^{(n)}$ radius of r_n
   c. Set $x^{(n+1)} = x^{(n)} + r_n \gamma_n$
3. Else, when $\text{distance}(x^{(n)} > \Gamma) \leq \varepsilon$
   a. $x_f =$ touching point at the boundary
   b. Return x_f as the estimator for x



Walk-on-Sphere on Infinite Domain

If we run Walk-on-Sphere on an infinite domain, the sphere will simply grow large, likely to diverge into infinity.



Method

Background

Given the **Poisson Equation**, and a **Coordinate Transform**

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \mathbf{x} \in \Omega$$

$$\phi : \Omega_{\text{inv}} \to \Omega, \phi(\cdot) = \phi^{-1}(\cdot) = \frac{\cdot}{|\cdot|^2}$$

Let $\mathbf{x} = \phi(\mathbf{y})$ what is the **Transformed Poisson Equation**?

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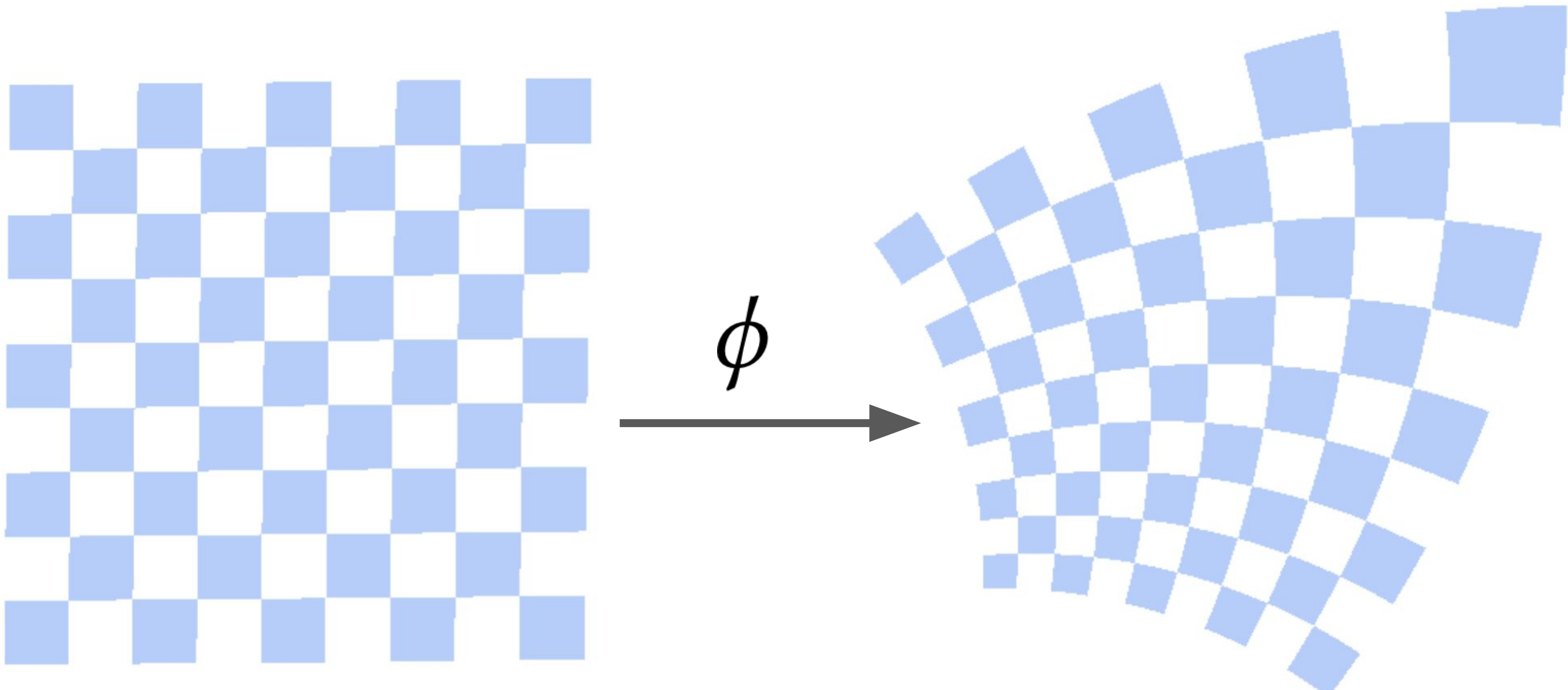
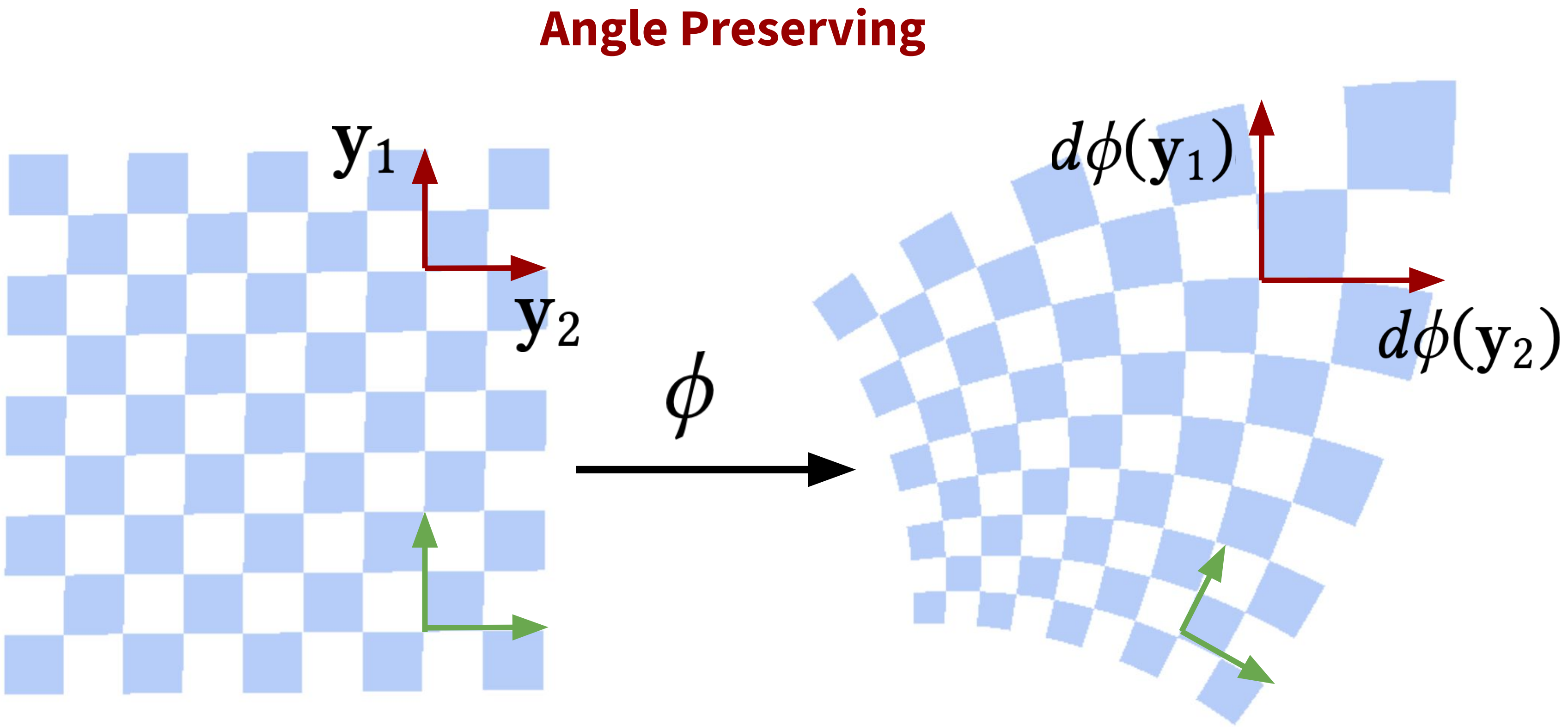
what is the transformed Δ ?

Background - Conformal Mapping

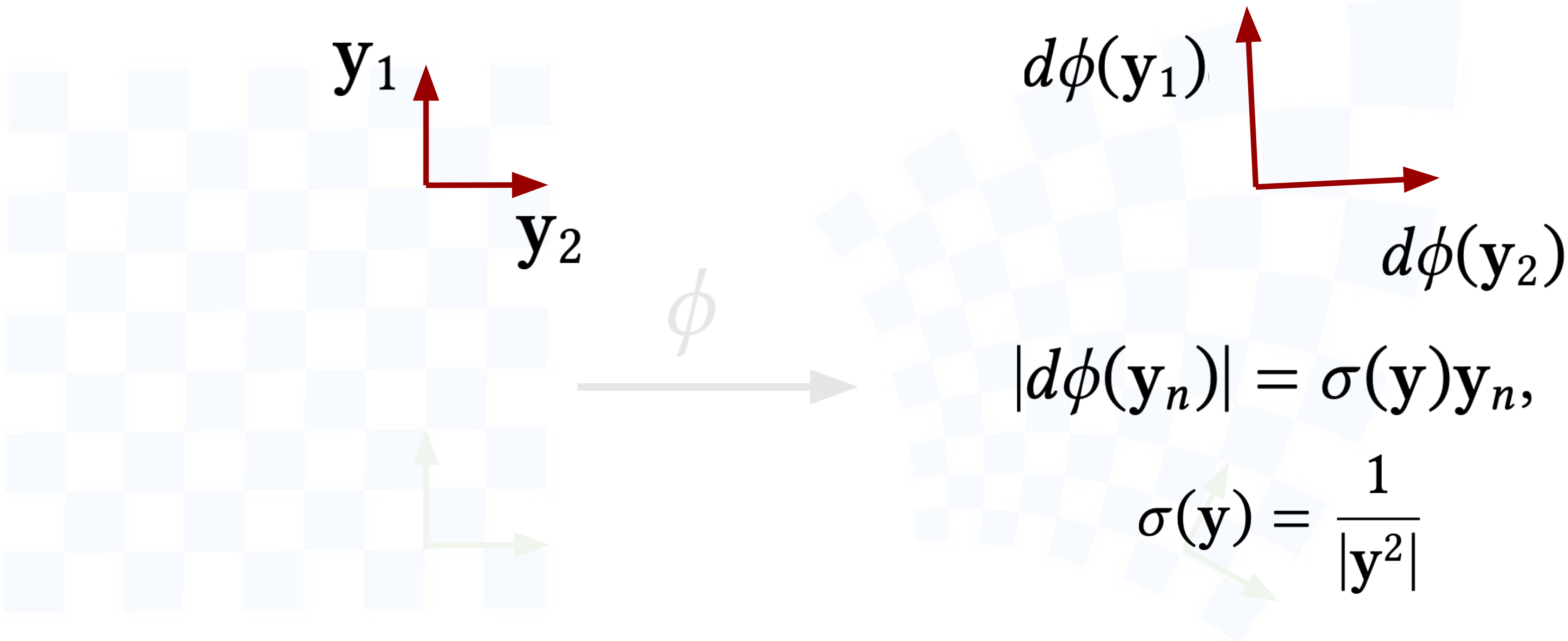


Background - Conformal Mapping



Background - Conformal Mapping

Angle Preserving

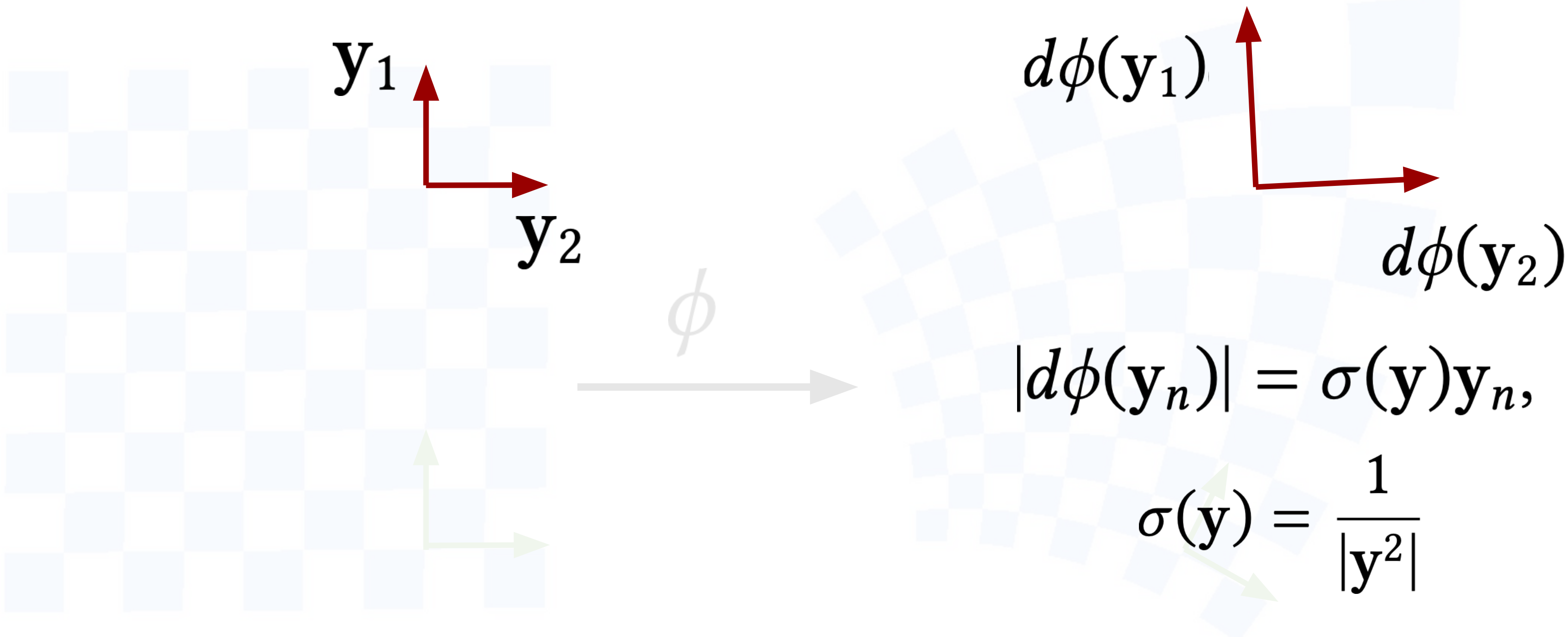


$$|d\phi(\mathbf{y}_n)| = \sigma(\mathbf{y})\mathbf{y}_n,$$

$$\sigma(\mathbf{y}) = \frac{1}{|\mathbf{y}^2|}$$

Background - Conformal Mapping

Angle Preserving



$$|d\phi(\mathbf{y}_n)| = \sigma(\mathbf{y})\mathbf{y}_n,$$

$$\sigma(\mathbf{y}) = \frac{1}{|\mathbf{y}^2|}$$

$$< d\phi(\mathbf{y}_1), d\phi(\mathbf{y}_2) >= \sigma(\mathbf{y})^2 < y_1, y_2 >$$

Background

Given the **Poisson Equation**, and a **Coordinate Transform**

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \mathbf{x} \in \Omega$$

$$\phi \,:\, \Omega_{\text{inv}} \to \Omega, \phi(\cdot) = \phi^{-1}(\cdot) = \frac{\cdot}{|\cdot|^2}$$

Let $\mathbf{x} = \phi(\mathbf{y})$ what is the **Transformed Poisson Equation**?

$$(\Delta^{\sigma} u)(\phi(\mathbf{y})) = f(\phi(y))$$

Background

$$U = u \circ \phi, F = f \circ \phi$$

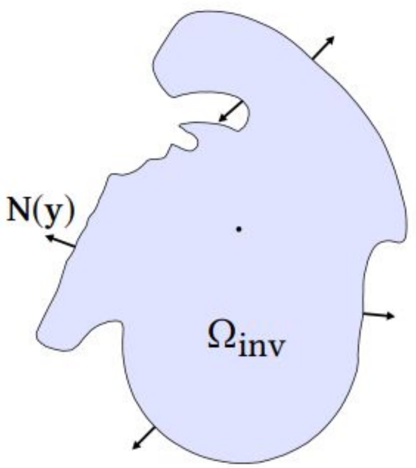
$$\Delta^\sigma U(\mathbf{y}) = |\mathbf{y}|^6 \nabla \cdot \left(\frac{1}{|\mathbf{y}|^2} \nabla U(\mathbf{y}) \right)$$

Let $\mathbf{x} = \phi(\mathbf{y})$ what is the **Transformed Poisson Equation**?

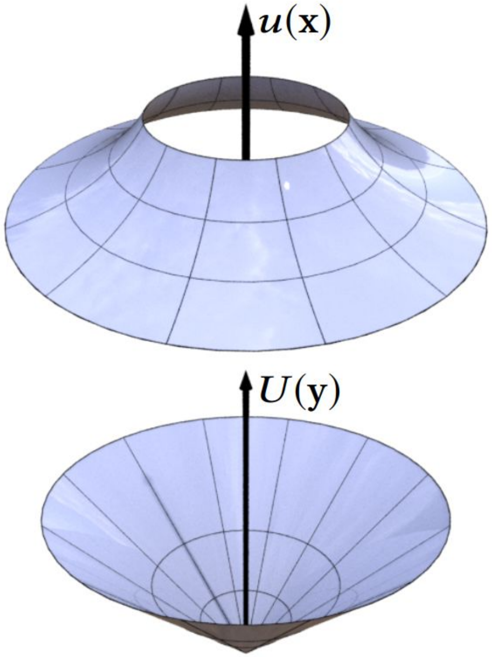
$$|\mathbf{y}|^6 \nabla \cdot \left(\frac{1}{|\mathbf{y}|^2} \nabla U(\mathbf{y}) \right) = F(\mathbf{y})$$

Domain Inversion - Problems

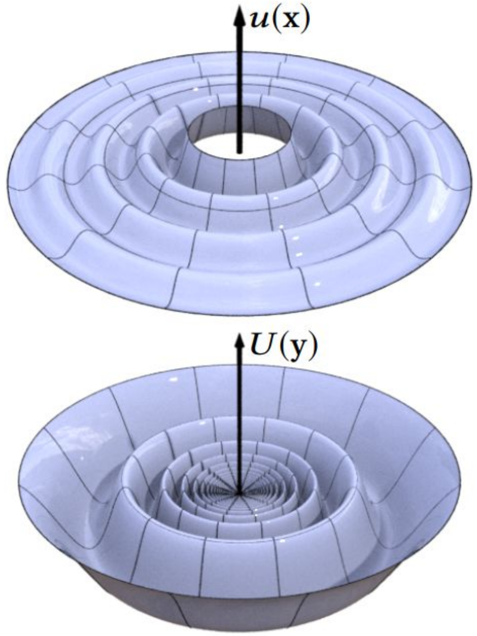
Domain is not compact.
Singularity at origin



Unsmoothed at origin



Does not work well when $\mathbf{U(y)}$ is not sufficiently well-behaved at origin



Kelvin Transformation

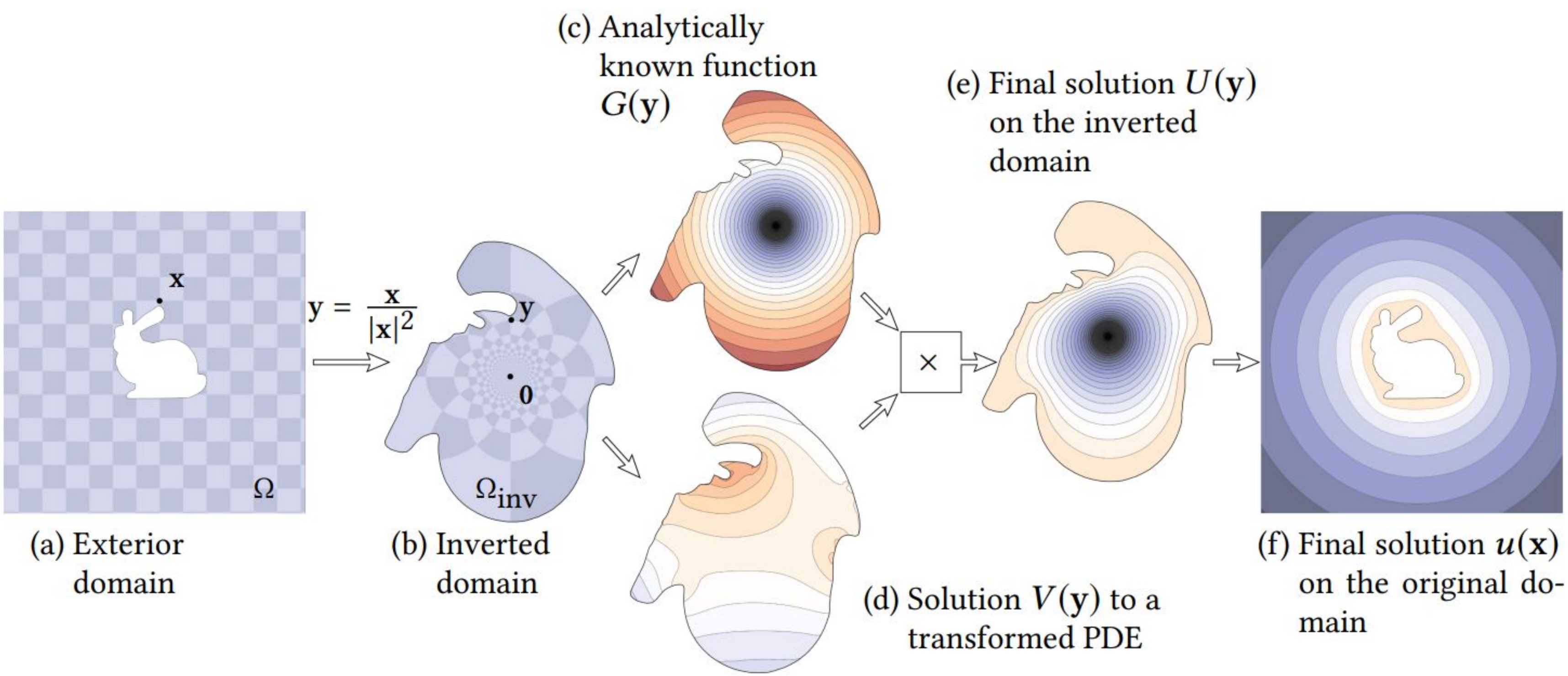
We have the following decomposition

$$U(\mathbf{y}) = G(\mathbf{y})V(\mathbf{y})$$

If we let $G(\mathbf{y}) = |\mathbf{y}|$, then the solution is found by solving

$$\Delta V(\mathbf{y}) = \frac{1}{|\mathbf{y}|^5} F(\mathbf{y}), \mathbf{y} \in \Omega_{\text{inv}}$$

Pipeline

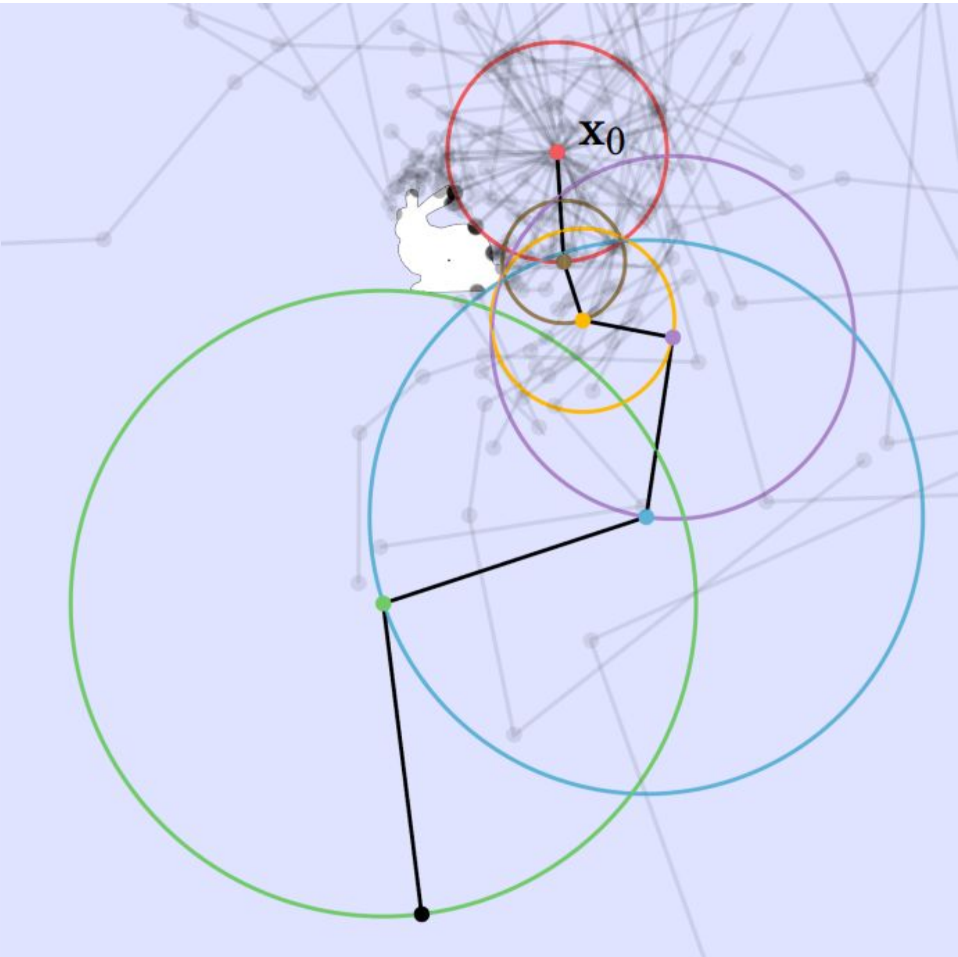


x
Ω
$y = \frac{x}{|x|^2}$
y
0
Ω_{inv}
(a) Exterior domain
(b) Inverted domain
(c) Analytically known function $G(\mathbf{y})$
(d) Solution $V(\mathbf{y})$ to a transformed PDE
×
(e) Final solution $U(\mathbf{y})$ on the inverted domain
(f) Final solution $u(\mathbf{x})$ on the original domain

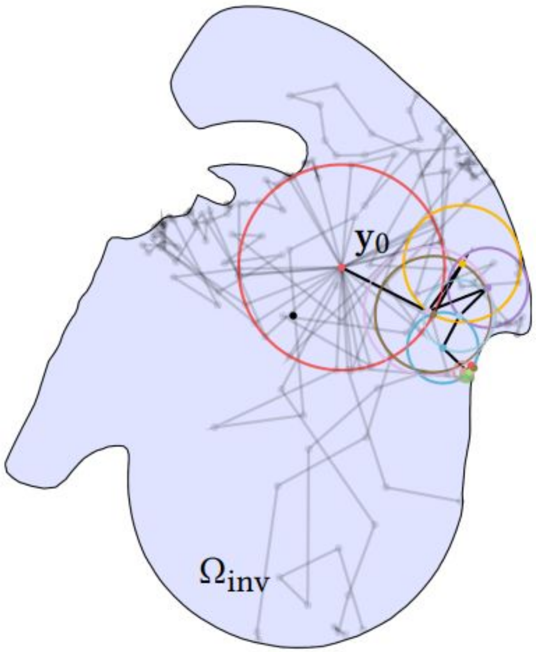
Experiment

Walk-on-Sphere with Russian Roulette

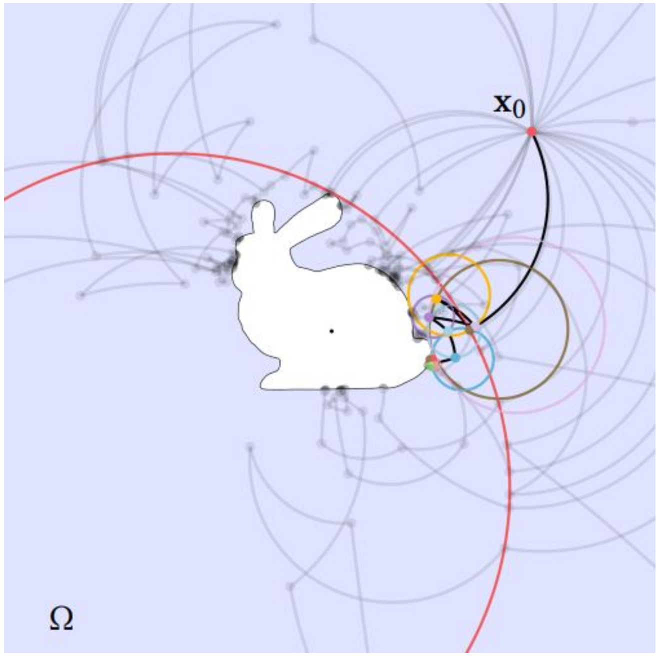
If we run Walk-on-Sphere on an infinite domain, the sphere will simply grow large, likely to diverge into infinity.



Walk-on-Sphere with Kelvin Transformation

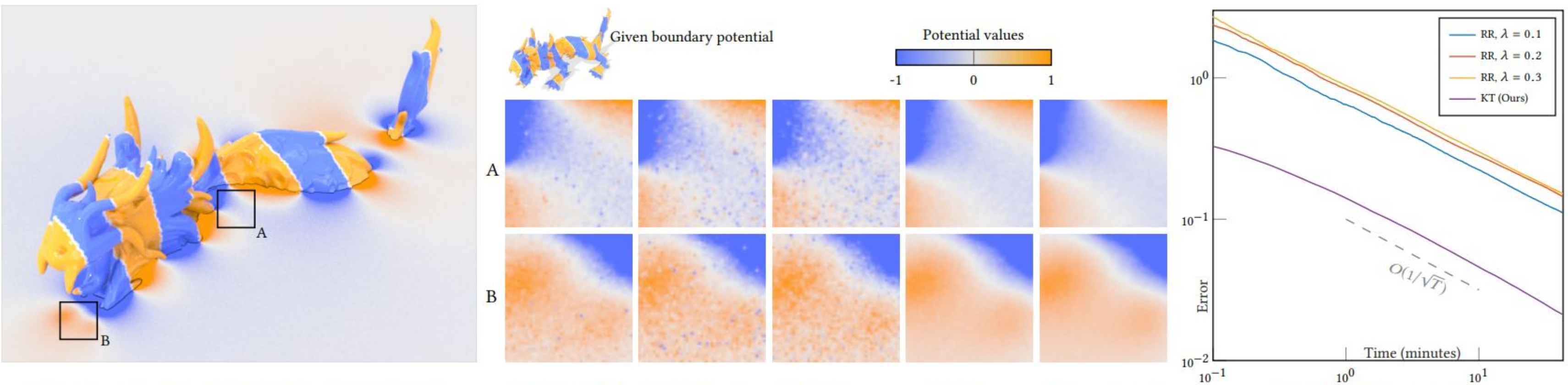


Inverted domain



Invert of inverted domain (original)

Error comparison - Laplace equation



(a) Laplace equation solved by Kelvin transform (KT).

(b) RR, $\lambda = 0.1$ Error = 11.04% (Equal time)

(c) RR, $\lambda = 0.2$ Error = 14.42% (Equal time)

(d) RR, $\lambda = 0.3$ Error = 15.12% (Equal time)

(e) **KT (Ours)** Error = 2.13% (40 minutes)

(f) Ground truth

(g) Laplace problem error plot.

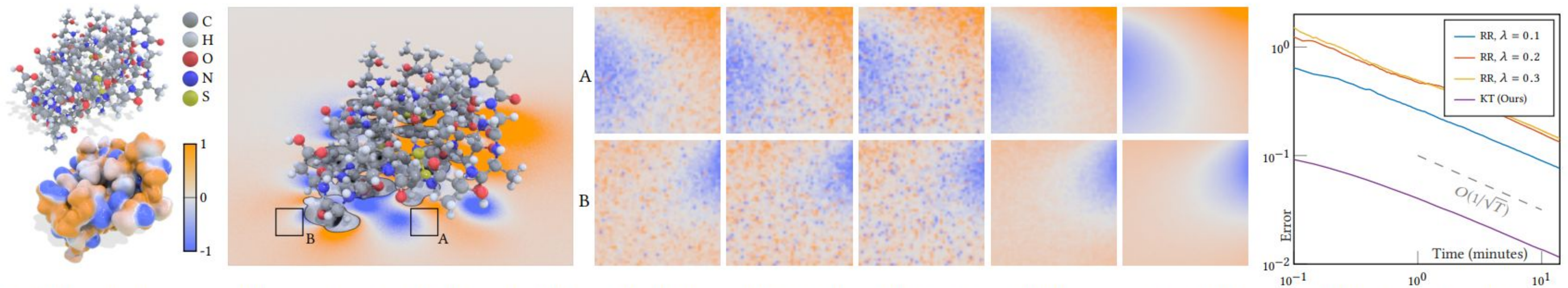
Russian Roulette

Kelvin transformation

Error plot
(Error - Time Graph)

To achieve equal quality result from RR, they needs 20 ~ 40 hours.

Error comparison - Electrostatic potential map



(a) 1CRN protein and its electrostatic potential map.

(b) Laplace equation solved by Kelvin transform (KT).

(c) RR, $\lambda = 0.1$ Error = 7.61% (Equal time)

(d) RR, $\lambda = 0.2$ Error = 13.32% (Equal time)

(e) RR, $\lambda = 0.3$ Error = 14.40% (Equal time)

(f) **KT (Ours)** Error = 1.16% (15 minutes)

(g) Ground truth

(h) Laplace problem error plot.

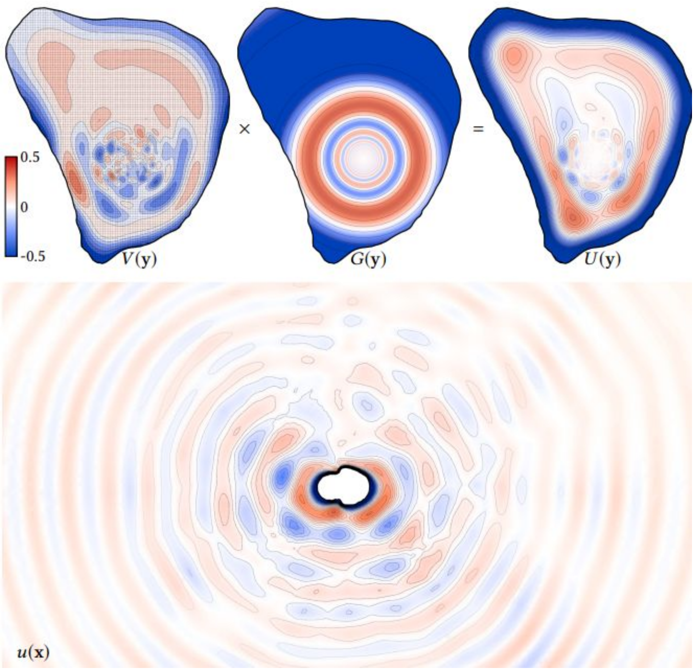
Error plot
(Error - Time Graph)

Russian Roulette

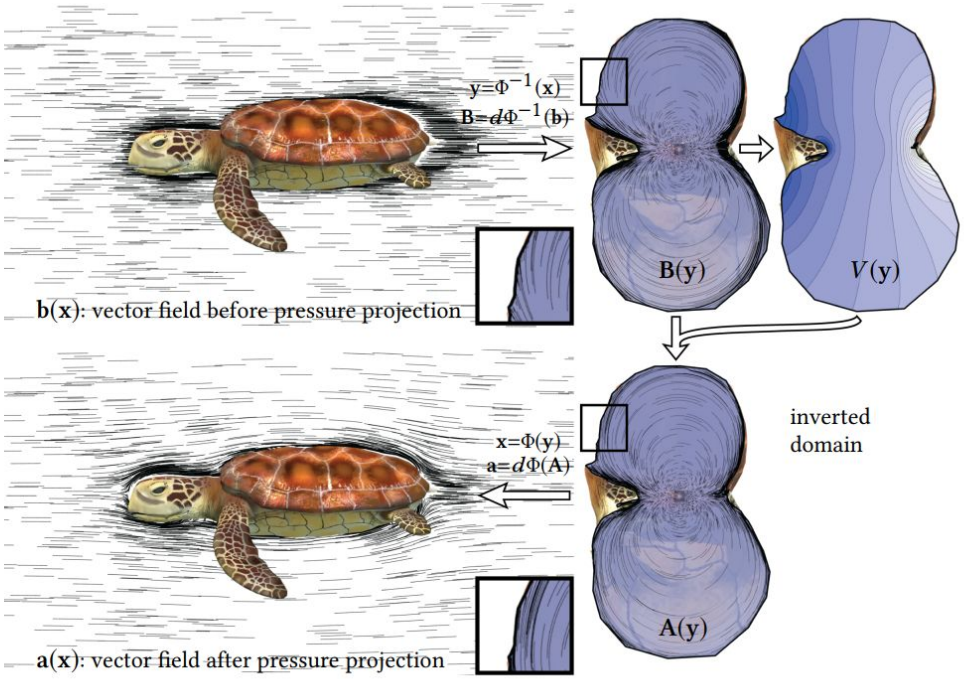
Kelvin transformation

To achieve equal quality result from RR, they needs 2 ~ 10 hours.

Pipeline to Result examples

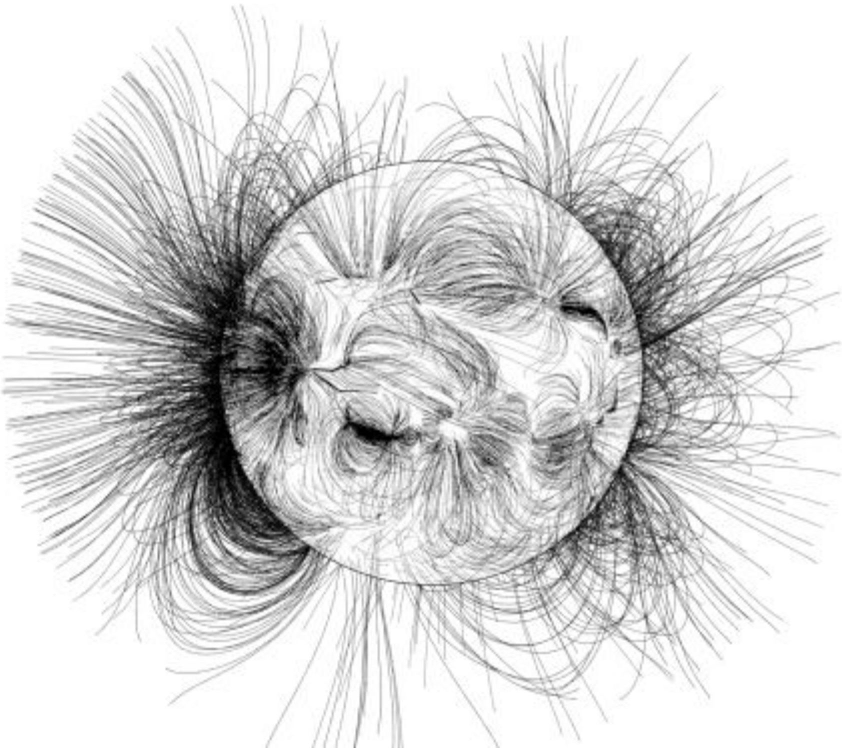


The Helmholtz equation
on an infinite domain

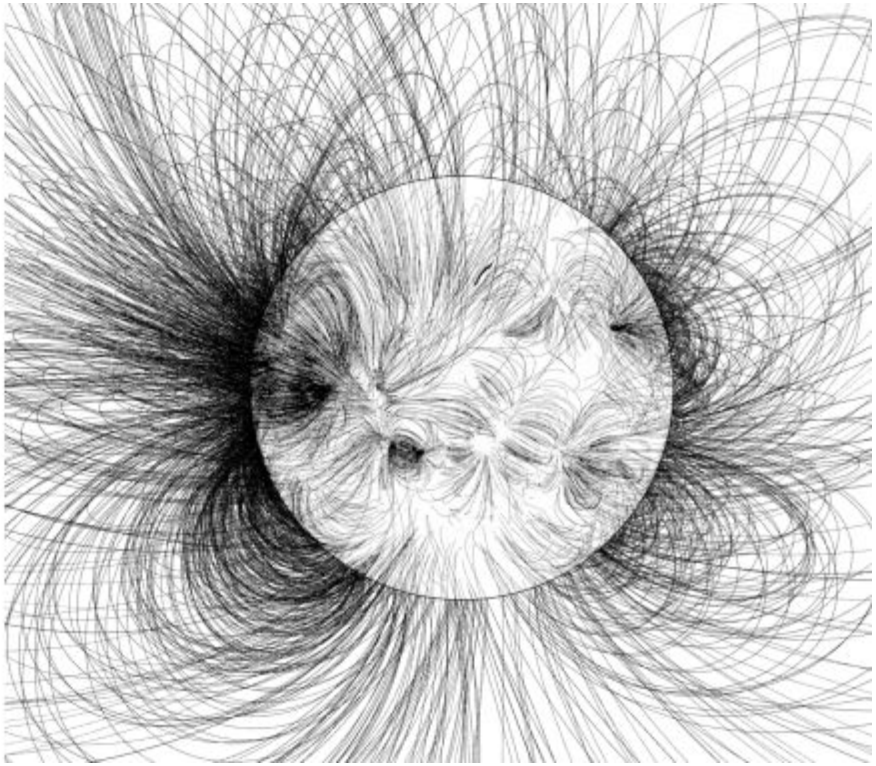


Pressure projection
on an infinite domain

Comparison with truncated domain



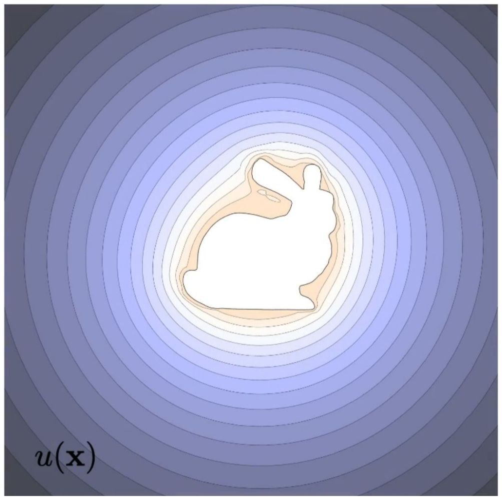
By domain truncation



By Kelvin transformation

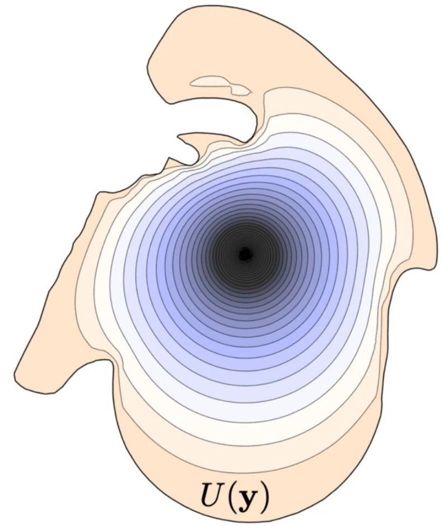
Main Takeaway

Summary



PDE problem
on an infinite domain

Kelvin
transformation



PDE problem
on an bounded domain

Takeaway

Wide range of applications

ex) WoS, Poisson problem, Helmholtz equation, …

Future work

- Not obvious how to generalize the Kelvin transform to PDEs of vector-valued functions or tensor valued-functions such as in continuum mechanics

Quiz

Does WoS with Kelvin Transform converges faster?

(a) True
(b) False

When we solve the PDE, do we solve the inverse problem, then invert the inverse solution to get the right solution?

(a) True
(b) False